PERGAMON

International Journal of Solids and Structures 37 (2000) 7043–7053

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Calculation of wave fields using elastodynamic reciprocity

Jan D. Achenbach *

Center for Quality Engineering and Failure Prevention, Northwestern University, 2137 Sheridan Road, Evanston, IL 60208, USA

Received 28 November 1999

Abstract

We consider axisymmetric time-harmonic wave motions generated by point-load excitation. Such problems are conventionally solved by the use of integral transform techniques. A typical example is the wave motion generated by a point force in an elastic layer. In this article, it is shown that, in a much simpler manner, the unknown modal coefficients for a superposition of wave modes can be conveniently obtained by the use of the Betti–Rayleigh reciprocity theorem. In this integral relation, which connects two elastodynamic states, one of the states is a wave mode of the actual wave field, whereas the other is an appropriately selected auxiliary solution. Simple expressions for the unknown coefficients quickly follow. The approach is also applied to the surface wave field generated by a time-harmonic normal point load on a half-space, where the amplitude constant of the generated surface wave motion is obtained. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Wave fields; Reciprocity; Normal load; Plate; Half-space

1. Introduction

Reciprocity theorems are among the classical results of acoustic, elastodynamic and electromagnetic theory. The various forms of reciprocity theorems, together with applications, have been discussed in some detail by de Hoop (1995). In general terms, a reciprocity theorem of the theory of elasticity provides an integral relation between two states of the same linearly elastic body. For the elastostatic case, the principal theorem is due to Betti (1872). A more general theorem, which includes the elastodynamic case, was given by Rayleigh (1873). In the present article, we consider the reciprocal theorem for time-harmonic elastodynamics, and we present what we believe to be a new application.

In the first part of this paper, the reciprocity theorem is used to determine the coefficients of wave mode expansions for elastic wave guides. The example that is considered is the wave motion produced in an elastic layer by a time harmonic point load applied normal to the faces of the layer. Point load problems are usually solved by an application of the Hankel transform technique, and an evaluation of the inverse transform by contour integration and residue calculus. That approach does in fact result in the displacement

* Tel.: +1-847-491-5527; fax: +1-847-491-5227.

E-mail address: achenbach@nwu.edu (J.D. Achenbach).

PII: S0020-7683(99)00325-X

expressions that are superpositions of wave modes. In this article, the displacements are directly expressed as wave mode expansions, and the coefficients are obtained by applying the reciprocity theorem to the wave mode expansions together with a suitably selected auxiliary solution.

The general formulation of the wave fields follows a recent article by Achenbach (1998). In a Cartesian coordinate system (x_1, x_2, z), the displacement components are expressed in terms of a function $\varphi(x_1, x_2)\exp(\mathrm{i}\omega t)$ which satisfies a simple reduced wave equation, and which acts as a carrier wave for propagation parallel to the x_1, x_2 plane. The carrier wave supports thickness motions that depend on the z coordinate only. This formulation is particularly suited for Lamb waves in a layer and surface waves on a half-space. For the specific problems considered in this article, Hankel functions represent the appropriate carrier waves. The analysis of the thickness motions results in the Rayleigh–Lamb frequency equation for the layer and the well-known Rayleigh equation for the velocity of surface waves on a half-space.

Numerous treatments of point source excitation of an elastic layer can be found in the technical literature. The review article on guided waves by Chimenti (1997) has a section on this topic which lists quite a large number of articles that are based on the application of integral transforms and/or numerical techniques, particularly, articles of a more recent origin and articles dealing with anisotropic plates. We also mention the work of Miklowitz (1962), Weaver and Pao (1982), Santosa and Pao (1989) and Vasudevan and Mal (1985). These articles are for transient loads, and they use integral intransforms, but the role of wave modes was discussed in some detail.

We also apply the reciprocity theorem to calculate the surface wave motion generated by the application of a point force to a half-space. It is of interest that the calculation does not include a consideration of the body waves generated by the point load. It is shown that the result is the same as obtained in the conventional manner by the use of the integral transform approach.

2. Formulation

A homogeneous, isotropic, linearly elastic material is referred to a Cartesian coordinate system with axes x_1, x_2 and z. Following a previous article by Achenbach (1998), we seek solutions for the displacement components in the general forms

$$u_\alpha(\boldsymbol{x}, t) = \frac{1}{k} V(z) \frac{\partial \varphi}{\partial x_\alpha}(x_1, x_2)\mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

$$u_z(\boldsymbol{x}, t) = W(z)\varphi(x_1, x_2)\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

where k is a wave-number-like quantity

$$k = \omega/c. \tag{3}$$

In the following analysis, the harmonic time factor $\exp(\mathrm{i}\omega t)$ will be omitted, and Greek indices will exclusively refer to the x_1 and x_2 axes.

Solutions of forms (1) and (2) were considered by Achenbach (1998), who showed that these expressions satisfy the elastodynamic equations of motion if the dimensionless function $\varphi(x_1, x_2)$ is taken as the solution of

$$\varphi_{,\beta\beta} + k^2\varphi = 0, \tag{4}$$

where repeated suffices indicate a summation, and $V(z)$ and $W(z)$ are solutions of the following system of ordinary differential equations:

$$(\lambda+\mu)\frac{\mathrm{d}W}{\mathrm{d}z}+\frac{\mu}{k}\frac{\mathrm{d}^2V}{\mathrm{d}z^2}+\frac{\rho\omega^2}{k}V(z)=k(\lambda+2\mu)V(z), \tag{5}$$

$$(\lambda+2\mu)\frac{\mathrm{d}^2W}{\mathrm{d}z^2}+\rho\omega^2W(z)=(\lambda+\mu)k\frac{\mathrm{d}V}{\mathrm{d}z}+\mu k^2W(z). \tag{6}$$

Here λ and μ are Lamé's constants and ρ is the mass density. Solutions of forms (1) and (2) are particularly convenient for bodies with one or more free surfaces or interfaces parallel to the x_1x_2 plane. Examples are Lamb waves in a layer or surface waves in a half-space. For such cases, the function $\varphi(x_1,x_2)$ acts as a carrier wave for the propagation in planes parallel to the x_1x_2 plane, while the functions $V(z)$ and $W(z)$ describe the thickness motions for Lamb waves and the depth variations for surface waves.

In this article, we will focus the attention on wave motions that are axially symmetric relative to the z axis. The relevant solution of Eq. (4) for an outgoing wave is then a Hankel function:

$$\varphi(r)=H_0^{(2)}(kr),\quad \text{where } r=(x_1^2+x_2^2)^{1/2}. \tag{7}$$

We can equally well consider an incoming wave, i.e., a wave which converges on $r=0$,

$$\varphi(r)=H_0^{(1)}(kr). \tag{8}$$

For the outgoing wave case, Eqs. (1) and (2) simplify to

$$u_r(r,z)=-V(z)H_1^{(2)}(kr), \tag{9}$$

$$u_z(r,z)=W(z)H_0^{(2)}(kr). \tag{10}$$

Expressions for the corresponding stresses can be obtained from Hooke's law as

$$\sigma_{rz}=\Sigma_{rz}(z)H_1^{(2)}(kr), \tag{11}$$

$$\sigma_{zz}=\Sigma_{zz}(z)H_0^{(2)}(kr), \tag{12}$$

$$\sigma_{rr}=\Sigma_{rr}(z)H_0^{(2)}(kr)+\overline{\Sigma}_{rr}(z)\frac{1}{r}H_1^{(2)}(kr), \tag{13}$$

where

$$\Sigma_{rz}(z)=-\mu\left[\frac{\mathrm{d}V}{\mathrm{d}z}+kW(z)\right], \tag{14}$$

$$\Sigma_{zz}(z)=-\mu\left[\frac{c_\mathrm{L}^2}{c_\mathrm{T}^2}\left(kV-\frac{\mathrm{d}W}{\mathrm{d}z}\right)-2kV\right], \tag{15}$$

$$\Sigma_{rr}(z)=-\mu\left[\frac{c_\mathrm{L}^2}{c_\mathrm{T}^2}\left(kV-\frac{\mathrm{d}W}{\mathrm{d}z}\right)+2\frac{\mathrm{d}W}{\mathrm{d}z}\right], \tag{16}$$

$$\overline{\Sigma}_{rr}(z)=2\mu V. \tag{17}$$

In these expressions, c_L and c_T are the velocities of longitudinal and transverse waves:

$$c_\mathrm{L}^2=\frac{\lambda+2\mu}{\rho}\quad\text{and}\quad c_\mathrm{T}^2=\frac{\mu}{\rho}. \tag{18a, b}$$

3. Reciprocity theorem

The reciprocity theorem relates two elastodynamic states of the same body. For two distinct time-harmonic states of the same frequency, labeled by superscripts A and B: $f_i^{\mathrm{A}}, u_i^{\mathrm{A}}, \sigma_{ij}^{\mathrm{A}}$, and $f_i^{\mathrm{B}}, u_i^{\mathrm{B}}, \sigma_{ij}^{\mathrm{B}}$, where f_i^{A} and f_i^{B} are body forces, we have for a region V with boundary S (Achenbach et al., 1982, p. 34)

$$\int_V \left(f_i^{\mathrm{A}} u_i^{\mathrm{B}} - f_i^{\mathrm{B}} u_i^{\mathrm{A}}\right) \mathrm{d}V = \int_S \left(u_i^{\mathrm{A}} \sigma_{ij}^{\mathrm{B}} - u_i^{\mathrm{B}} \sigma_{ij}^{\mathrm{A}}\right) n_j \, \mathrm{d}S, \tag{19}$$

where n_j are the components of the outward normal.

Let us now consider a circular domain of an elastic layer defined by $|z| \leqslant h,\ r \leqslant b$ with two states, both of which satisfy $\sigma_{zz}(\pm h) = \sigma_{zr}(\pm h) \equiv 0$, but where state A is the solution to a point load of magnitude P applied in the z direction at $x_1 = x_2 \equiv 0, z = 0$, and state B is an axially symmetric nonsingular solution in the domain of interest. After integrating over the polar angle θ, Eq. (19) yields

$$Pu_z^{\mathrm{B}}(0) = 2\pi b \int_{-h}^{h} \left\{ \left[u_r^{\mathrm{A}} \sigma_{rr}^{\mathrm{B}} - u_r^{\mathrm{B}} \sigma_{rr}^{\mathrm{A}}\right] + \left[u_z^{\mathrm{A}} \sigma_{rz}^{\mathrm{B}} - u_z^{\mathrm{B}} \sigma_{rz}^{\mathrm{A}}\right] \right\} \mathrm{d}z, \tag{20}$$

where all terms in the integrand are defined at $r = b$.

The reciprocity theorem becomes useful as an aid in problem solving if we can select an appropriate nonsingular solution for state B. In this article, we will explore such auxiliary solutions of the following general forms:

$$u_r^{\mathrm{B}} = -\tfrac{1}{2} V^{\mathrm{B}}(z) \left[H_1^{(2)}(kr) + H_1^{(1)}(kr)\right], \tag{21}$$

$$u_z^{\mathrm{B}} = \tfrac{1}{2} W^{\mathrm{B}}(z) \left[H_0^{(2)}(kr) + H_0^{(1)}(kr)\right]. \tag{22}$$

The displacements in Eqs. (21) and (22), which are the sums of an outgoing and an incoming wave, are bounded at $r = 0$. It can be verified that the left-hand side of Eq. (20) becomes

$$Pu_z^{\mathrm{B}}(0) = PW^{\mathrm{B}}(0). \tag{23}$$

To evaluate the right-hand side, we introduce the expressions given by Eqs. (9)–(11), (13), (14) and (16), (17) for state A, and (21) and (22) and the corresponding stresses for state B. After some manipulation, Eq. (20) reduces to

$$\begin{aligned} PW^{\mathrm{B}}(0) = {} & \pi b (I_{\mathrm{AB}} - I_{\mathrm{BA}}) H_0^{(2)}(kb) H_1^{(2)}(kb) \\ & + \pi b I_{\mathrm{AB}} H_1^{(1)}(kb) H_0^{(2)}(kb) - \pi b I_{\mathrm{BA}} H_1^{(2)}(kb) H_0^{(1)}(kb), \end{aligned} \tag{24}$$

where

$$I_{\mathrm{AB}} = \int_{-h}^{h} \left(\Sigma_{rr}^{\mathrm{A}}(z) V^{\mathrm{B}}(z) + \Sigma_{rz}^{\mathrm{B}}(z) W^{\mathrm{A}}(z)\right) \mathrm{d}z, \tag{25}$$

and I_{BA} follows from Eq. (25) by interchanging A and B. Next, it will be shown that Eq. (24) can be used to actually express the fields radiating from a point force.

4. Point force on a layer

The theory of the preceding sections will now be used to determine expressions for the wave motion in an elastic layer, defined by $|z| \leqslant h, 0 \leqslant r < \infty$, which is subjected to a time-harmonic point load directed along the z axis, and applied at $z = 0$, i.e., in the mid-plane of the layer. Clearly the elastodynamic displacement fields will be antisymmetric relative to the plane $z = 0$, and axisymmetric with respect to the z axis.

Wave motion in an elastic layer can be studied in terms of an infinite number of wave modes that may be separated into symmetric and antisymmetric modes relative to the mid-plane of the layer. For a given value of the circular frequency, ω, the wave number, k_n, is one of the many solutions of the Rayleigh–Lamb frequency equation. For displacement solutions of the general forms given by Eqs. (9) and (10), the wave modes have been stated by Achenbach (1998). For the antisymmetric modes, we have

$$V^n(z) = a_1 \sin(pz) + a_2 \sin(qz), \tag{26}$$

$$W^n(z) = a_3 \cos(pz) + a_4 \cos(qz), \tag{27}$$

where

$$a_1 = 2\sin(qh), \qquad a_2 = -[(k_n^2 - q^2)/k_n^2]\sin(ph), \tag{28a, b}$$

$$a_3 = 2(p/k_n)\sin(qh), \qquad a_4 = [(k_n^2 - q^2)/qk_n]\sin(ph). \tag{29a, b}$$

In these expressions

$$p^2 = \frac{\omega^2}{c_L^2} - k_n^2, \tag{30a}$$

$$q^2 = \frac{\omega^2}{c_T^2} - k_n^2. \tag{30b}$$

The corresponding expressions for the stresses are

$$\Sigma_{rz}^n(z) = \mu[a_5 \cos(pz) + a_6 \cos(qz)], \tag{31}$$

$$\Sigma_{zz}^n(z) = \mu[a_7 \sin(pz) + a_8 \sin(qz)], \tag{32}$$

$$\Sigma_{rr}^n(z) = \mu[a_9 \sin(pz) + a_{10} \sin(qz)], \tag{33}$$

where

$$a_5 = -4p\sin(qh), \qquad a_6 = -\left[(k_n^2 - q^2)^2/qk_n^2\right]\sin(ph), \tag{34a, b}$$

$$a_7 = [2(k_n^2 - q^2)/k_n]\sin(qh), \qquad a_8 = -[2(k_n^2 - q^2)/k_n]\sin(ph), \tag{35a, b}$$

$$a_9 = [2(2p^2 - k_n^2 - q^2)/k_n]\sin(qh), \qquad a_{10} = [2(k_n^2 - q^2)/k_n]\sin(ph). \tag{36a, b}$$

It may be checked that $\Sigma_{zz}^h(h) = 0$ is identically satisfied, whereas $\Sigma_{rz}^n(h) = 0$ yields the Rayleigh–Lamb frequency equation for antisymmetric modes

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{(k_n^2 - q^2)^2}{4pqk_n^2}. \tag{37}$$

For a specific value of ω, Eq. (37) is an equation for k_n. For each solution k_n, Eqs. (26) and (27) and (31)–(33) define a specific mode, i.e., a set of displacements and stresses that satisfy the equations of motion and the boundary conditions.

Intuitively, it is now to be expected that, at least at some distance from the applied load, the displacements generated by the point load, applied in the mid-plane of the layer, can be expressed as a summation over antisymmetric modes:

$$u_r^{\mathrm{P}} = -\sum_{m=0}^{\infty} D_m[a_1 \sin(pz) + a_2 \sin(qz)]H_1^{(2)}(k_m r), \tag{38}$$

$$u_z^{\mathrm{P}} = \sum_{m=0}^{\infty} D_m[a_3 \cos(pz) + a_4 \cos(qz)]H_0^{(2)}(k_m r), \tag{39}$$

where a_1, a_2, a_3 and a_4 are defined by Eqs. (28a,b) and (29a,b) and k_m are the solutions of Eq. (37) for specified ω. These solutions represent outgoing waves, but it remains to determine the constants D_m such that u_r^{P} and u_z^{P} correspond to the prescribed loads. In the remainder of this paper, we propose a simple method to determine the constants.

Now, let us return to the reciprocity relation represented by Eq. (19). For the two states, A and B, we select two modes: mode m with wave number k_m of Eqs. (38) and (39), and mode n with wave number k_n of the auxiliary solution given by Eqs. (21) and (22). Instead of I_{AB} we then have

$$I_{mn} = \int_{-h}^{h} [\Sigma_{rr}^m(z)V^n(z) + \Sigma_{rz}^n(z)W^m(z)]\,\mathrm{d}z. \tag{40}$$

It has been shown by Achenbach and Xu (1998) that

$$I_{mn} \equiv 0 \quad \text{for } m \neq n, \tag{41}$$

whereas for $m = n$ we have

$$I_{nn} = \mu[c_1 \sin^2(ph) + c_2 \sin^2(qh)], \tag{42}$$

where

$$c_1 = \frac{(k_n^2 - q^2)(k_n^2 + q^2)}{2q^3 k_n^3}[2qh(k_n^2 - q^2) + (k_n^2 + 7q^2)\sin(2qh)], \tag{43}$$

$$c_2 = \frac{k_n^2 + q^2}{pk_n^3}[4k_n^2 ph - 2(k_n^2 - 2p^2)\sin(2ph)]. \tag{44}$$

Inspection of Eq. (24) now shows that by virtue of Eq. (41), the right hand vanishes completely for $m \neq n$. For $m = n$, the first term vanishes and Eq. (24) becomes

$$PW^n(0) = \pi b I_{nn}\left[H_1^{(1)}(k_n b)H_0^{(2)}(k_n b) - H_1^{(2)}(k_n b)H_0^{(1)}k_n b)\right]D_n. \tag{45}$$

Using the following identity for Hankel functions (McLachlan, 1961, p. 198)

$$H_v^{(1)}(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}H_v^{(2)}(\xi) - \frac{\mathrm{d}}{\mathrm{d}\xi}H_v^{(1)}(\xi)H_v^{(2)}(\xi) = -\frac{4\mathrm{i}}{\pi\xi} \tag{46}$$

for $v = 0$ and $\xi = k_n b$, Eq. (45) yields

$$D_n = \frac{k_n}{4\mathrm{i}}\frac{PW^n(0)}{I_{nn}}, \tag{47}$$

where I_{nn} is defined by Eq. (42).

The expansions for the wave motion given by Eqs. (38) and (39) can easily be generalized to a point load applied at an arbitrary position in the interior of the layer, by splitting the problem in a symmetric and an antisymmetric problem.

5. Point force on a half-space

The expressions given by Eqs. (9)–(13) are also very suitable for surface waves. For a half-space $z \geqslant 0$ we seek solutions of Eqs. (5) and (6) of the general forms that decay exponentially with depth. Using Eqs. (9)–(13), such solutions have been derived by Deutsch et al. (1999) as

$$V^R(z) = d_1 e^{-pz} + d_2 e^{-qz}, \tag{48}$$

$$W^R(z) = d_3 e^{-pz} - e^{-qz}, \tag{49}$$

where

$$p^2 = k^2 - \frac{\omega^2}{c_L^2} \quad \text{and} \quad q^2 = k^2 - \frac{\omega^2}{c_T^2}, \tag{50a, b}$$

$$d_1 = -\frac{1}{2}\frac{k^2+q^2}{kp}, \qquad d_2 = \frac{q}{k}, \tag{51a, b}$$

$$d_3 = \frac{1}{2}\frac{k^2+q^2}{k^2}. \tag{52}$$

It can be verified that Eqs. (48) and (49) satisfy the system of ordinary differential equations (5) and (6). The corresponding stress terms follow from Eqs. (14)–(16) as

$$\Sigma^R_{rr}(z) = \mu[d_4 e^{-pz} + d_5 e^{-qz}], \tag{53}$$

$$\Sigma^R_{rz}(z) = \mu[d_6 e^{pz} + d_7 e^{-qz}], \tag{54}$$

$$\Sigma^R_{zz}(z) = \mu[d_8 e^{-pz} + d_9 e^{-qz}], \tag{55}$$

where

$$d_4 = \frac{1}{2}(k^2+q^2)\frac{2p^2+k^2-q^2}{pk^2}, \quad d_5 = -2q, \tag{56a, b}$$

$$d_6 = -\frac{k^2+q^2}{k}, \qquad d_7 = \frac{k^2+q^2}{k}, \tag{57a, b}$$

$$d_8 = -\frac{1}{2}\frac{(k^2+q^2)^2}{pk^2}, \qquad d_9 = 2q. \tag{58a, b}$$

The surface $z = 0$ should be free of surface tractions. It is immediately seen that $\Sigma^R_{rz}(0) = 0$, whereas $\Sigma^R_{zz}(0) = 0$ requires that

$$F(k) = (k^2+q^2)^2 - 4k^2pq = 0. \tag{59}$$

By substituting p and q from Eqs. (50a, b) and using $\omega = kc$, Eq. (59) assumes the better known form

$$\left(2 - \frac{c^2}{c_T^2}\right)^2 - 4\left(1 - \frac{c^2}{c_L^2}\right)^{1/2}\left(1 - \frac{c^2}{c_T^2}\right)^{1/2} = 0. \tag{60}$$

Eq. (60) is the equation for the phase velocity, $c = c_R$, of Rayleigh surface waves. Thus, as is well known, along the surface of a half-space only one mode of surface wave motion, with wave number $k_R = \omega/c_R$, can propagate.

It is now tempting to apply Eq. (24) to the problem of a time-harmonic normal point load applied on the surface of the half-space. A point load generates surface waves, and we might consider the possibility of determining the amplitude of these surface waves by using Eq. (24), with the modification that the integration in I_{AB}, Eq. (25), should be taken from $-\infty$ to 0. For state A we take

$$u_r^A = -CV^R(z)H_1^{(2)}(k_R r), \tag{61}$$

$$u_z^A = CW^R(z)H_0^{(2)}(k_R r), \tag{62}$$

where C is an unknown amplitude factor. For state B we take the dummy solution

$$u_r^B = -\tfrac{1}{2}V^R(z)\left[H_1^{(2)}(k_R r) + H_1^{(1)}(k_R r)\right], \tag{63}$$

$$u_z^B = \tfrac{1}{2}W^R(z)\left[H_0^{(2)}(k_R r) + H_0^{(1)}(k_R r)\right], \tag{64}$$

Eq. (24) now yields

$$PW^R(0) = -\frac{4i}{k}CI_{AB} \quad \text{or} \quad C = -\frac{k}{4i}\frac{PW^R(0)}{I_{AB}}, \tag{65}$$

where Eq. (46) has been used, and I_{AB} is defined as

$$I_{AB} = \int_0^\infty \left[\Sigma_{rr}^R(z)V^R(z) + \Sigma_{rz}^R(z)W^R(z)\right] dz. \tag{66}$$

Substitution of the expressions for $\Sigma_{rr}^R(z), \Sigma_{rz}^R(z), V^R(z)$ and $W^R(z)$, given by Eqs. (53), (54) and (48) and (49), into Eq. (66) yields a relatively simple integral over z which can easily be evaluated. The parameter k_R appearing in the integral is the solution of Eq. (59). It is well known that the equation $F(k) = 0$ has one real valued positive solution. To simplify further manipulation, we introduce the dimensionless Rayleigh wave velocity by

$$\xi = \frac{\omega}{c_T}\frac{1}{k_R}. \tag{67}$$

We also introduce

$$q_R^2 = 1 - \xi^2, \tag{68}$$

$$p_R^2 = 1 - \frac{\xi^2}{\kappa^2}, \tag{69}$$

where

$$\kappa^2 = \frac{c_L^2}{c_T^2} = \frac{2(1-\upsilon)}{1-2\upsilon}, \tag{70}$$

where υ is Poisson's ratio. Carrying out the integration of Eq. (66) yields after some further manipulation

$$J_{AB} = -\frac{1+3q_R^2}{2q_R} + \left(1+q_R^2\right)\left[\frac{1+q_R^2}{2p_R} + \frac{q_R}{p_R}(p_R - q_R) + \frac{1}{p_R}\right] - \frac{\left(1+q_R^2\right)^2}{8p_R^3}\left(1 + 4p_R^2 - q_R^2\right), \tag{71}$$

where

$$I_{AB} = \mu J_{AB}. \tag{72}$$

At the surface, $z = 0$, the vertical displacement due to the Rayleigh wave now follows from Eqs. (62) and (65) as

$$u_z = -\frac{k_\mathrm{R}}{4\mathrm{i}}\frac{P(d_3-1)^2}{I_\mathrm{AB}}H_0^{(2)}(k_\mathrm{R}r). \tag{73}$$

This expression is rewritten as

$$\frac{\mu}{P}\frac{u_z}{k_\mathrm{R}} = U_z(\xi)H_0^{(2)}(k_\mathrm{R}r), \tag{74}$$

where

$$U_z(\xi) = \frac{\mathrm{i}}{16}\frac{(1-q_\mathrm{R}^2)^2}{J_\mathrm{AB}}, \tag{75}$$

and J_AB is defined by Eq. (71).

It is of interest to compare this expression with the corresponding one obtained by the conventional technique using the integral transform approach. Achenbach (1973, p. 310) has used the Laplace and Hankel transforms to determine the response of a half-space to a normal point load of arbitrary time dependence. For a time-harmonic point load, the Hankel transform of the vertical displacement at $z = 0$ follows from Achenbach (1973, p. 313, Eq. (7.233)) by replacing the Laplace transform parameter p by $\mathrm{i}\omega$

$$u_z^H = -\frac{P}{2\pi}\frac{1}{\mu}\frac{\omega^2}{c_\mathrm{T}^2}\frac{p(k)}{F(k)}, \tag{76}$$

where $p(k)$ and $F(k)$ are defined by Eqs. (50a) and (59), respectively. By application of the inverse Hankel transform, we find

$$u_z = -\frac{P}{2\pi}\frac{1}{\mu}\frac{\omega^2}{c_\mathrm{T}^2}\int_0^\infty \frac{p(k)}{F(k)}J_0(kr)k\,\mathrm{d}k. \tag{77}$$

To obtain the Rayleigh wave contribution from this integral, we use

$$\begin{aligned} J_0(kr) &= \tfrac{1}{2}H_0^{(2)}(kr) + \tfrac{1}{2}H_0^{(1)}(kr) \\ &= \tfrac{1}{2}H_0^{(2)}(kr) - \tfrac{1}{2}H_0^{(2)}(-kr). \end{aligned} \tag{78}$$

Substitution of this result into Eq. (77) allows the integral to be rewritten as

$$u_z = -\frac{P}{4\pi}\frac{1}{\mu}\frac{\omega^2}{c_\mathrm{T}^2}\int_{-\infty}^\infty \frac{p(k)}{F(k)}kH_0^{(2)}(kr)\,\mathrm{d}k. \tag{79}$$

The Rayleigh wave is the contribution from the pole at the point $k = k_\mathrm{R}$, where k_R is the solution of Eq. (59), i.e., $F(k) = 0$. We find

$$u_z = \frac{P}{2}\frac{\mathrm{i}}{\mu}\frac{\omega^2}{c_\mathrm{T}^2}\frac{(k_\mathrm{R}^2-\omega^2/c_\mathrm{L}^2)^{1/2}}{F'(k_\mathrm{R})}k_\mathrm{R}H_0^{(2)}(k_\mathrm{R}r), \tag{80}$$

where

$$F'(k_\mathrm{R}) = \left.\frac{\mathrm{d}F(k)}{\mathrm{d}k}\right|_{k=k_\mathrm{R}}. \tag{81}$$

Recasting Eq. (80) in the form of Eq. (74), we find

$$U_z(\xi) = \frac{i\xi^2(1-\xi^2/\kappa^2)^{1/2}}{8F'(\xi)}, \tag{82}$$

where ξ is defined by Eq. (67), and

$$F'(\xi) = 2(2-\xi^2) - 2p_R q_R - \frac{q_R^2 + p_R^2}{p_R q_R}. \tag{83}$$

Eqs. (75) and (82) both represent the surface wave generated by a time-harmonic point load applied normal to the surface of a homogeneous, isotropic, linearly elastic half-space. The question of interest is whether they do in fact give the same numerical result. This can easily be checked by a simple calculation. Let us consider the case of Poisson's ratio $\upsilon = 0.25$. For this case, Eq. (70) yields $\kappa^2 = 3$, and the solution of $F(k) = 0$ is

$$\xi = 0.919402. \tag{84}$$

Substitution of this value into Eqs. (75) and (82) yields for both cases exactly the same value, namely,

$$U_z = -0.917429i. \tag{85}$$

The author believes that it is in fact possible to manipulate the two expressions for U_z and shows analytically that they are the same.

6. Concluding comments

It has been shown in this article that the elastodynamic reciprocity theorem in conjunction with an appropriately selected auxiliary elastodynamic solution provides a relatively simple way to obtain displacement waves generated by point loads. For an elastic layer, the method yields modal expansions for the wave field generated by a time-harmonic point load applied normal to the faces of the layer. For an elastic half-space the surface wave generated by a time-harmonic point load normal to the surface, is obtained.

The approach presented in this article has recently been extended by Achenbach and Xu (1999) to the analysis of wave motion in a layer generated by a time-harmonic point load of arbitrary orientation.

Acknowledgements

This work was carried out during the course of research sponsored by the Office of Naval Research under contract no. N00014-89-J-1362.

References

Achenbach, J.D., 1973. Wave Propagation in Elastic Solids, Elsevier Science, Amsterdam.

Achenbach, J.D., 1998. Lamb waves as thickness vibrations superimposed on a membrane carrier wave. J. Acoust. Soc. Am. 103, 2283–2286.

Achenbach, J.D., Xu, Y., 1999. Wave motion in an elastic layer generated by a time-harmonic point load of arbitrary direction. J. Acoust. Soc. Am., 106, 83–90.

Achenbach, J.D., Gautesen, A.K., McMaken, H., 1982. Ray Methods for Waves in Elastic Solids. Pitman Advanced Publishing Program, Boston.

Betti, E., 1872. Teori della elasticita. Il Nuovo Ciemento (Series 2), 7–10.

Chimenti, D.E., 1997. Guided waves in plates and their use in materials characterization. Appl. Mech. Rev. 50, 247–284.

de Hoop, A.T., 1995. Handbook of Radiation and Scattering of Waves. Academic Press, London.

Deutsch, W.A.K., Cheng, A., Achenbach, J.D., 1999. Focusing of Rayleigh waves: simulation and experiment. IEEE Trans. Ultrason. Ferroelect. Freq. Contr. 46, 333–340.

McLachlan, N.W., 1961. Bessel Functions for Engineers. Oxford University Press, London.

Miklowitz, J., 1962. Transient compressional waves in an infinite elastic plate or elastic layer overlying a rigid half-space. J. Appl. Mech. 29, 53–60.

Rayleigh, L., 1873. Some general theorems relating to vibration. Lond. Math. Soc. Proc. 4, 366–368.

Santosa, F., Pao, Y.-H., 1989. Transient axially symmetric response of an elastic plate. Wave Motion 11, 271–296.

Vasudevan, N., Mal, A.K., 1985. Response of an elastic plate to localized transient sources. J. Appl. Mech. 52, 356–362.

Weaver, R.L., Pao, Y.-H., 1982. Axisymmetric elastic waves excited by a point source in a plate. J. Appl. Mech. 49, 821–836.